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REVIEW ARTICLE

Solution of fractional-order differential equations based on the operational matrices of new fractional Bernstein functions

M.H.T. Alshbool [a,*], A.S. Bataineh [b], I. Hashim [a], Osman Rasit Isik [c]

[a] School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia
[b] Department of Mathematics, Faculty of Science, Al Balqa Applied University, 19117 Salt, Jordan
[c] Department of Mathematics, Faculty of Sciences, Mugla Sitki Kocman University, 48000 Mugla, Turkey

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Abstract An algorithm for approximating solutions to fractional differential equations (FDEs) in a modified new Bernstein polynomial basis is introduced. Writing $x \to x^\alpha$ $(0 < \alpha < 1)$ in the operational matrices of Bernstein polynomials, the fractional Bernstein polynomials are obtained and then transformed into matrix form. Furthermore, using Caputo fractional derivative, the matrix form of the fractional derivative is constructed for the fractional Bernstein matrices. We convert each term of the problem to the matrix form by means of fractional Bernstein matrices. A basic matrix equation which corresponds to a system of fractional equations is utilized, and a new system of nonlinear algebraic equations is obtained. The method is given with some priori error estimate. By using the residual correction procedure, the absolute error can be estimated. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

Contents

* Corresponding author.
E-mail address: alshbool.mohammed@gmail.com (M.H.T. Alshbool).
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1. Introduction

Fractional differential equations have several applications in mathematical physics, fluid flow, engineering and other areas of applications (Miller and Ross, 1993; Podlubny, 1999; Jafari and Momani, 2007; Daftardar and Jafari, 2007; Abdulaziz et al., 2008). In this paper, we consider the fractional differential equations (FDEs) of the form:

$$D^{\alpha}y(x) = g(x) + q(x)y(x) + z(x)(y(x))^{r}, \quad 0 \leqslant x \leqslant 1. \tag{1}$$

under the conditions

$$y(0) = \beta_1, \quad y(1) = \beta_2. \tag{2}$$

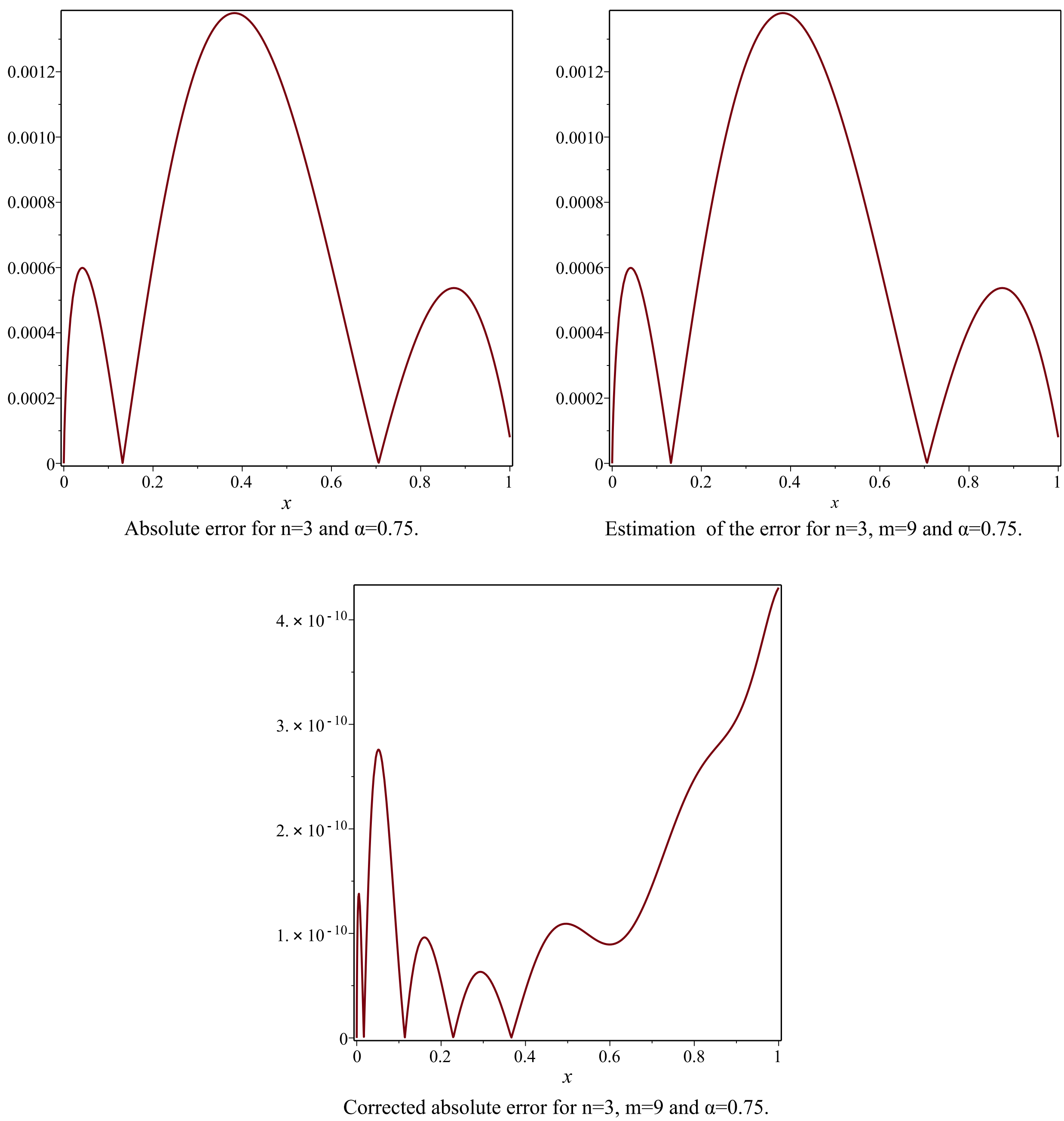


Figure 1 The absolute error, the estimated absolute error and the corrected absolute error to Example 1, for the case $n = 3, m = 9$ and $\alpha = 0.75$.

Here, $y(x)$ is an unknown function; $g(x)$, $q(x)$ and $z(x)$ are the functions that are defined in [0, 1].

Numerical and analytical methods are applied to solve fractional differential equation such as homotopy perturbation method (Hosseinnia et al., 2008), Modification of homotopy perturbation methods (Odibat and Momani, 2008), Taylor collocation method (Cenesiz et al., 2010), Jacobi operational matrix method (Kazem, 2013), Variational iteration method (Odibat and Momani, 2006), Legendre functions method (Kazem et al., 2013), Tau methods (Rad et al., 2014), A new operational matrix method (Saadatmandi and Dehghan, 2010). Yang et al. (2015) presented local fractional derivative operators and local integral transforms to solve fractional differential equations. Rostamya et al. (2013) proposed anew efficient basis to solve fractional partial differential equations.

Fractional Riccati differential equations have been solved by many efficient techniques such as Adomian's decomposition method (Gejji and Jafari, 2007; Momani and Shawagfeh, 2006) and new spectral wavelets methods (Abd-Elhameed and Youssri, 2014).

One of the most important numerical methods for solving linear and nonlinear equations is Bernstein polynomials which

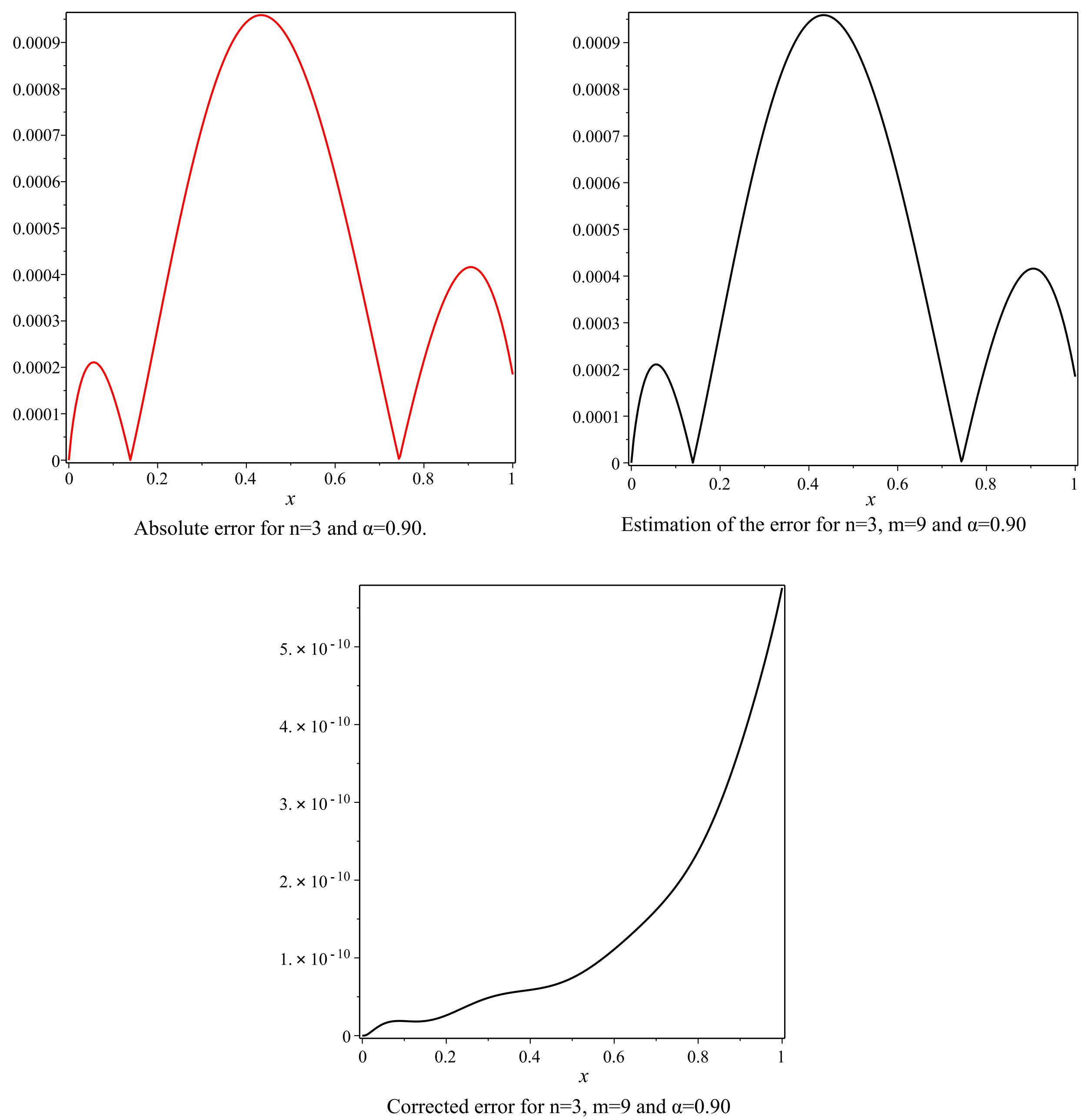


Figure 2 The absolute error, the estimated absolute error and the corrected absolute error to Example 1, for the case $n = 3, m = 9$ and $\alpha = 0.90$.

have been used by many authors. Pandey and Kumar (2012) used Bernstein operational matrix of differentiation to solve Lane–Emden type equation. Besides that Bhatti and Bracken (2007) solved the differential equations by using Galerkin method based on the Bernstein polynomial basis. Yousefi and Behroozifar (2010) presented an operational matrix method based on Bernstein polynomials for the differential equations. Recently, Yiming et al. (2014) solved a variable order time fractional diffusion equation using Bernstein polynomials. Alshbool and Hashim (2016) presented multistage Bernstein polynomials which is a new modification of Bernstein polynomials to find the solutions of fractional order stiff systems.

In the present paper, we utilize a new operational matrices method based on the Bernstein polynomials to solve linear and non-linear fractional differential equations. We generalize r for $r \in \mathbb{N}_0$, also the absolute error is estimated and corrected, as well as the addition of the roots of Chebyshev polynomials is used to reduce the interpolation error. The suggested method shows that this approach can solve the problem effectively.

This article is structured as follows. In Section 2, the definitions and properties of the fractional calculus are given. In

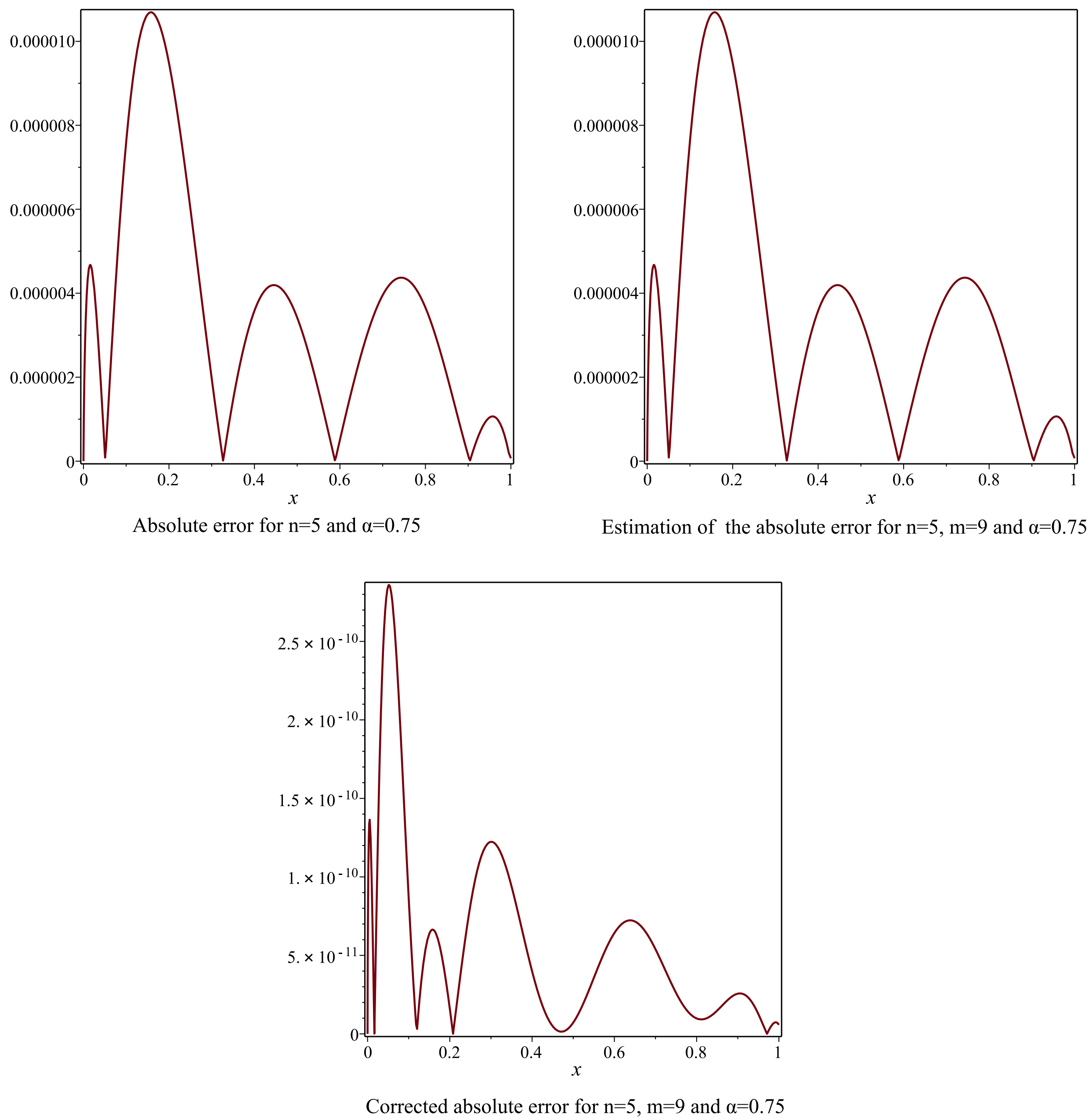


Figure 3 The absolute error, the estimated absolute error and the corrected absolute error to Example 1, for the case $n = 5, m = 9$ and $\alpha = 0.75$.

Section 3, our method and its applications are explained. In Section 4, an error analysis of the method and estimation of the error are presented, as well as the corrected absolute error. In Section 5, our numerical findings, exact solution and demonstration of the validity, accuracy and applicability of the operational matrices by considering numerical examples are reported. Section 6, consists of brief summary and conclusion.

2. Preliminaries and notations

In this section, we provide some definitions and properties of the fractional calculus (Diethelm et al., 2005; Podlubny, 1999).

Definition 2.1. A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \Re$, if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator (J^α) of order $\alpha \geqslant 0$, of a function $f \in C_\mu, \mu \geqslant -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(x) ds \quad (\alpha > 0),$$
$$J^0 f(x) = f(x), \tag{3}$$

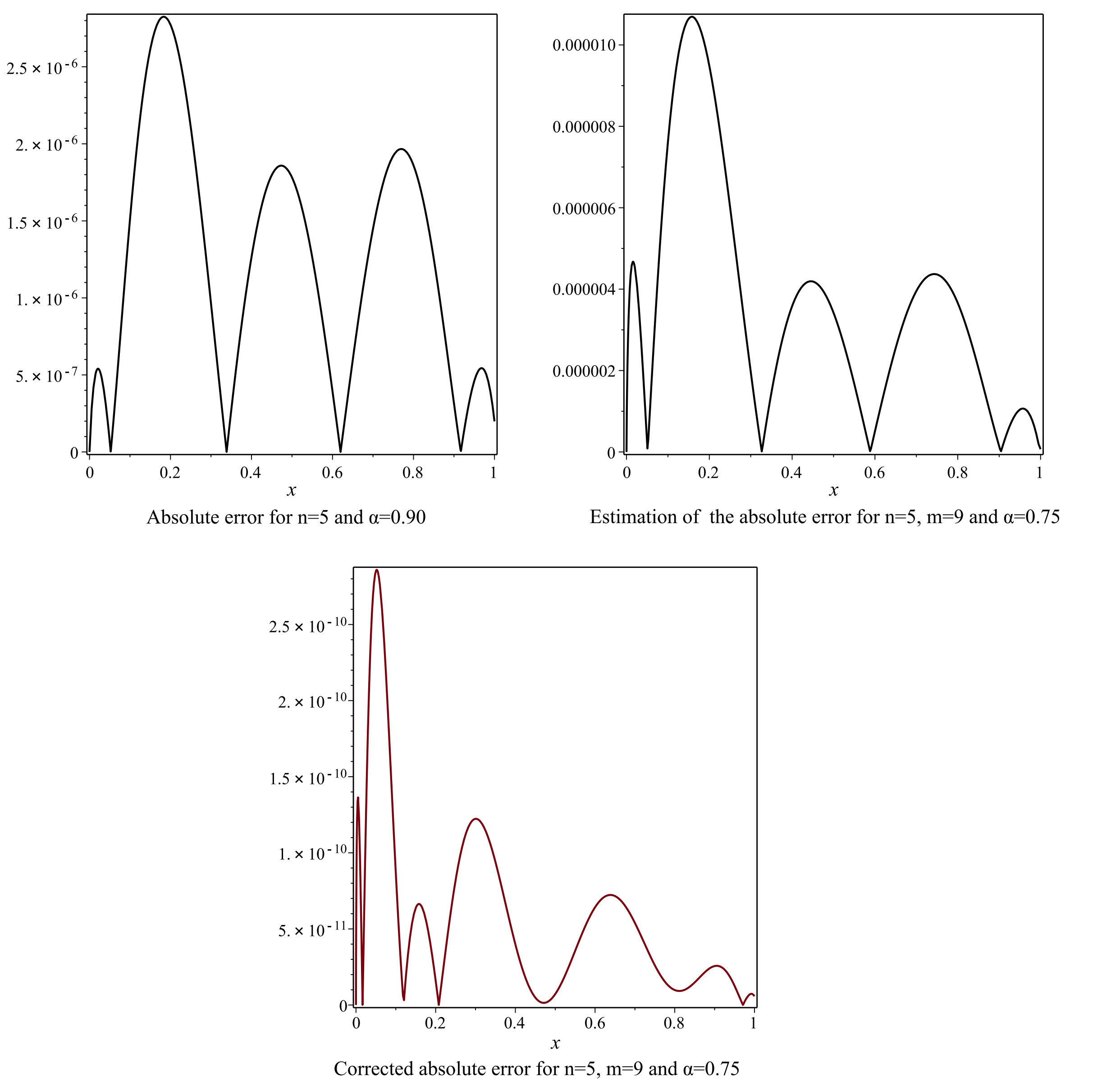
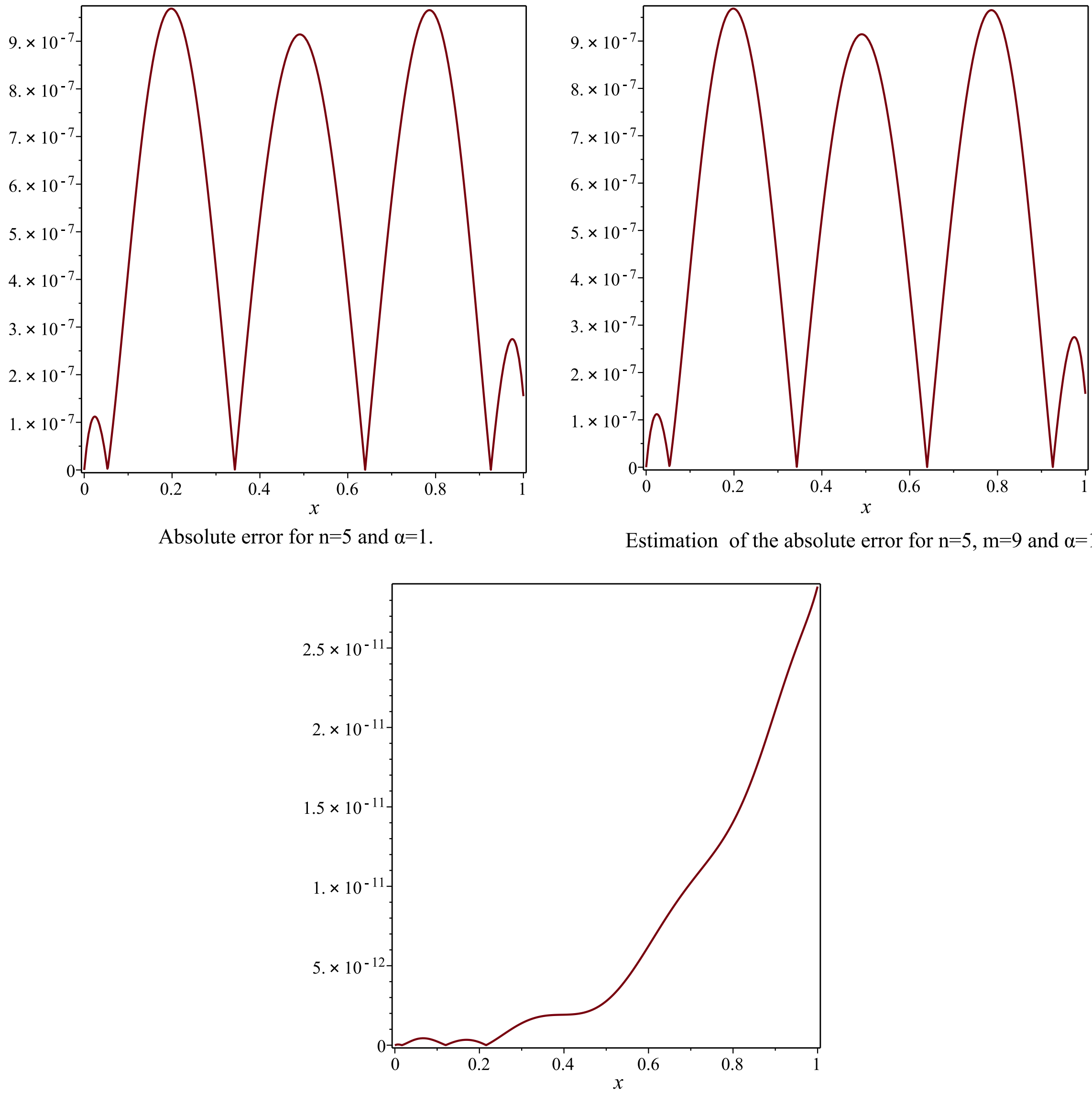


Figure 4 The absolute error, the estimated absolute error and the corrected absolute error to Example 1, for the case $n = 5, m = 9$ and $\alpha = 0.90$.



Corrected absolute error for n=5, m=9 and α=1.

Figure 5 The absolute error, the estimated absolute error and the corrected absolute error to Example 1, for the case $n = 5, m = 9$ and $\alpha = 1$.

where $\Gamma(\alpha)$ is well-known gamma function. Some of the properties of the operator J^{α}, which we will need here, are as follows: For $f \in C_{\mu}, \mu \geqslant -1, \alpha, \beta \geqslant 0$ and $\gamma \geqslant -1$:

1. $J^{\alpha}J^{\beta}f(x) = J^{\alpha+\beta}f(x)$,
2. $J^{\alpha}J^{\beta}f(x) = J^{\beta}J^{\alpha}f(x)$,
3. $J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$.

Definition 2.3. The fractional derivative (D^{α}) of $f(x)$, in the Caputo sense is defined as

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (x-s)^{n-\alpha-1}f^{(n)}(x)ds, \tag{4}$$

for $n-1 < \alpha < n, n \in N, x > 0, f \in C_{-1}^{n}$.

The following are two basic properties of the Caputo fractional derivative (Miller and Ross, 1993):

1. Let $f \in C_{-1}^{n}, n \in N$. Then $D^{\alpha}f, 0 \leqslant \alpha \leqslant n$ is well defined and $D^{\alpha}f \in C_{-1}$.
2. Let $n-1 \leqslant \alpha \leqslant n, n \in N$ and $f \in C_{\mu}^{n}, \mu \geqslant -1$. Then

$$(J^{\alpha}D^{\alpha})f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^{+})\frac{x^k}{k!}. \tag{5}$$

For the Caputo derivative we have

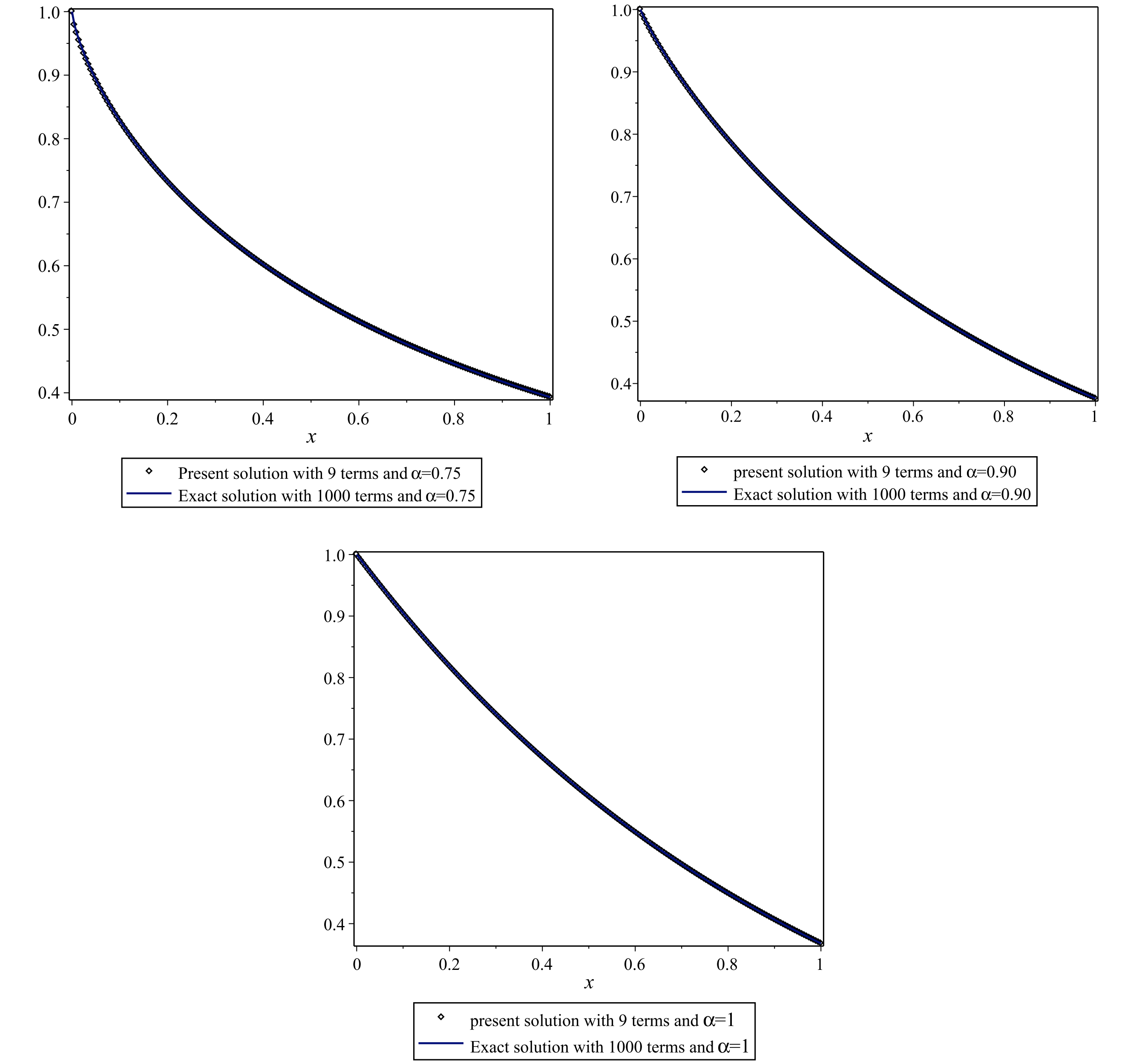


Figure 6 Compare between our present solution with terms 9 and Exact solution with 1000 terms.

$$D_*^\alpha c = 0, \quad (c \text{ constant}), \tag{6}$$

$$D_*^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in N_0 \text{ and } \beta \geqslant \lceil \alpha \rceil \text{ or } \beta > \lfloor \alpha \rfloor. \end{cases} \tag{7}$$

We note that the approximate solutions will be found by using the Caputo fractional derivative and its properties in this study.

3. Description of the method

To solve (1), (2) by developing the Bernstein polynomial approximation with the help of the matrix operations, the collocation method and the caputo fractional derivative are used.

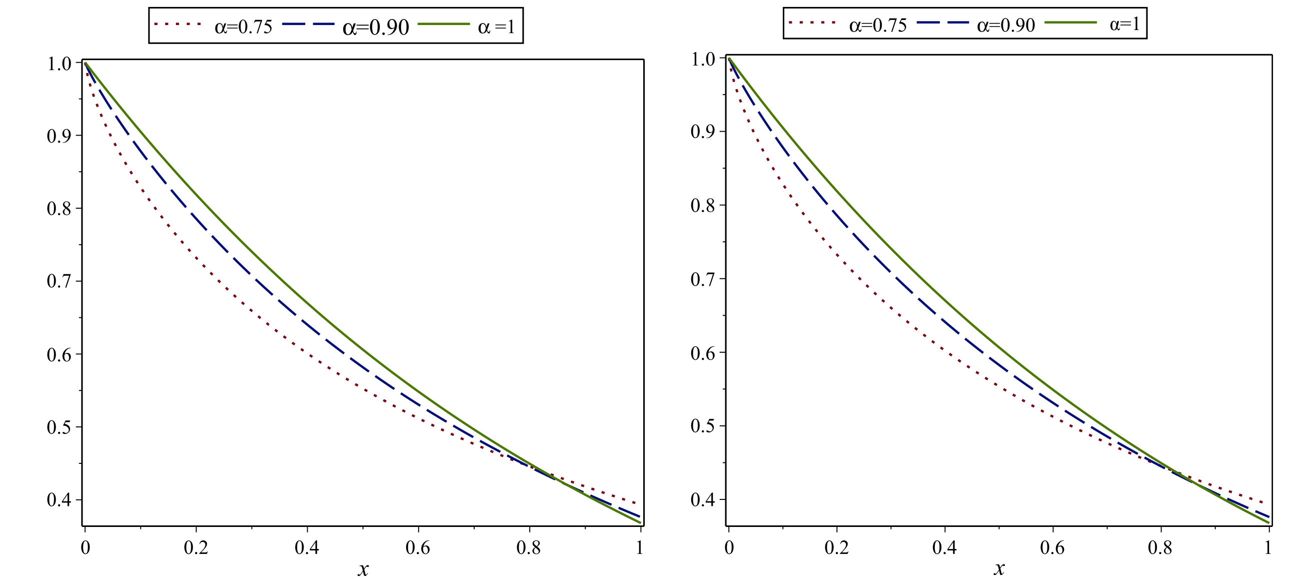
We obtain an approximate solution of the problem (1), (2) in the form of

$$y_{n,\alpha}(x) = \sum_{k=0}^{n} c_k B_{n,k}^\alpha (x - c). \tag{8}$$

Here $0 < alpha < 1, c_k, k = 0, 1, 2, \ldots, n$ are the unknown Bernstein coefficients, n is chosen for any positive integers, and $B_{k,n}^\alpha(x)$ are obtained by putting $x \to x^\alpha$ in Bernstein polynomials as defined by

$$B_{k,n}(x) = \binom{n}{k} \frac{x^k (R - x)^{n-k}}{R^n}, \quad k = 0, 1, 2, \ldots, n \quad x \in [0, R],$$
$$i = 0, 1, \ldots, n$$

it becomes



Compare between α=0.75,α=0.90 and α=1 to Example 1, for the case n=3

Compare between α=0.75,α=0.90 and α=1 to Example 1, for the case n=5

Figure 7 Compare between $\alpha = 0.75, \alpha = 0.90$ and $\alpha = 1$ to Examples 1, for the case $n = 3$, and $n = 5$.

$$B^{\alpha}_{k,n}(x) = \binom{n}{k}\frac{x^{k\alpha}(R - x^{\alpha})^{n-k}}{R^{n}}, \quad k = 0,1,2,\ldots,n \quad x \in [0,R], \quad i = 0,1,\ldots,n \tag{9}$$

so that

$$y_{n,\alpha}(x) = \sum_{k=0}^{n} c_k B^{\alpha}_{n,k}(x - c) = C^{T}\phi^{*}(x). \tag{10}$$

where

$$\phi^{*}(x) = [B^{\alpha}_{0,n}(x), B^{\alpha}_{1,n}(x), \ldots, B^{\alpha}_{n,n}(x)]^{\mathrm{T}}, \quad \mathrm{C}^{\mathrm{T}} = [\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_n]. \tag{11}$$

Here, the expression $(\phi^{*}(x))$ in (11) can be written as

$$\phi^{*}(x) = AX \tag{12}$$

Where

$$A = \begin{pmatrix} a_{00} & a_{01} & \ldots & a_{0n} \\ a_{10} & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n0} & a_{n1} & \ldots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ x^{\alpha} \\ x^{2\alpha} \\ \vdots \\ x^{n\alpha} \end{pmatrix}. \tag{13}$$

and

$$a_{ij} = \begin{cases} \frac{(-1)^{j-i}}{R^{j}}\binom{n}{i}\binom{n-i}{j-i}, & i \leqslant j \\ 0, & i > j \end{cases}$$

So that

$$A^{-1}\phi^{*}(x) = X. \tag{14}$$

By using the Caputo fractional derivative (7) and applying it on Eq. (12), the αth order fractional derivative of $\phi^{*}(x)$ can be written as

$$D^{\alpha}\phi^{*}(x) = A\frac{d^{\alpha}}{dx^{\alpha}}X. \tag{15}$$

$$D^{\alpha}\phi^{*}(x) = AX^{*}. \tag{16}$$

where

$$X^{*} = \begin{pmatrix} 0 \\ \Gamma(\alpha+1) \\ \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}x^{\alpha} \\ \vdots \\ \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}x^{(n-1)\alpha} \end{pmatrix}.$$

We can write (16) as

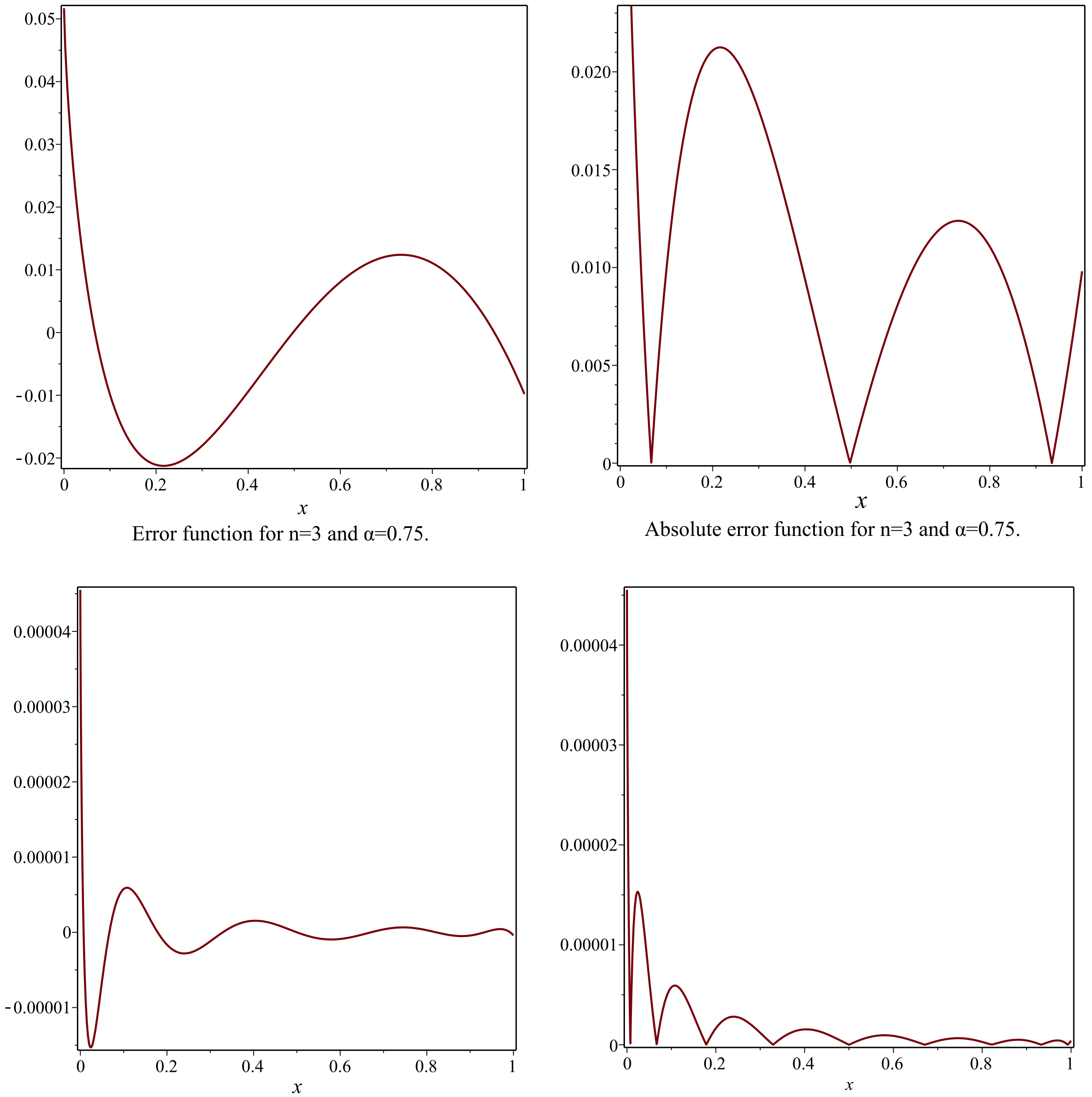
$$D^{\alpha}\phi^{*}(x) = A\Omega\overline{X}. \tag{17}$$

Where

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} \end{pmatrix}, \quad \overline{X} = \begin{pmatrix} 0 \\ 1 \\ x^{\alpha} \\ \vdots \\ x^{(n-1)\alpha} \end{pmatrix}. \tag{18}$$

Similarly, the D^{α} fractional derivative of $\phi(x)$ is given by the recurrence relation

$$D^{\alpha}\phi^{*}(x) = A\Omega\Psi X. \tag{19}$$



Error function for n=3 and α=0.75.

Absolute error function for n=3 and α=0.75.

Corrected of error function for n=3, m=9 and α=0.75.

Absolute Corrected of error function for n=3, m=9 and α=0.75.

Figure 8 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 2, for the case $n = 3, m = 9$ and $\alpha = 0.75$.

Where

$$\Psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We place relation (14) into Eq. (19) and then we get

$$D^{\alpha}\phi^{*}(x) = A\Omega\Psi A^{-1}\phi^{*}(x). \quad (20)$$

Finally, we obtain D^{α} as

$$D^{\alpha} = A\Omega\Psi A^{-1}. \quad (21)$$

Note that every matrix is in the dimension of $(n+1)\times(n+1)$.

Substituting (D^{α}) in Eq. (10) we obtain

$$D^{\alpha}y(x) \approx C^{T}D^{\alpha}\phi^{*}(x). \quad (22)$$

To obtain the solution of (1), the methods that are applied are:

Firstly, Eq. (10) is applied to approximate $(y_{n,\alpha}(x))^{r}$ and $g(x)$ as

$$(y_{n,\alpha}(x))^{r} = (C^{T}\phi^{*}(x))^{r} \quad (23)$$

$$g(x) \approx G^{T}\phi^{*}(x), \quad (24)$$

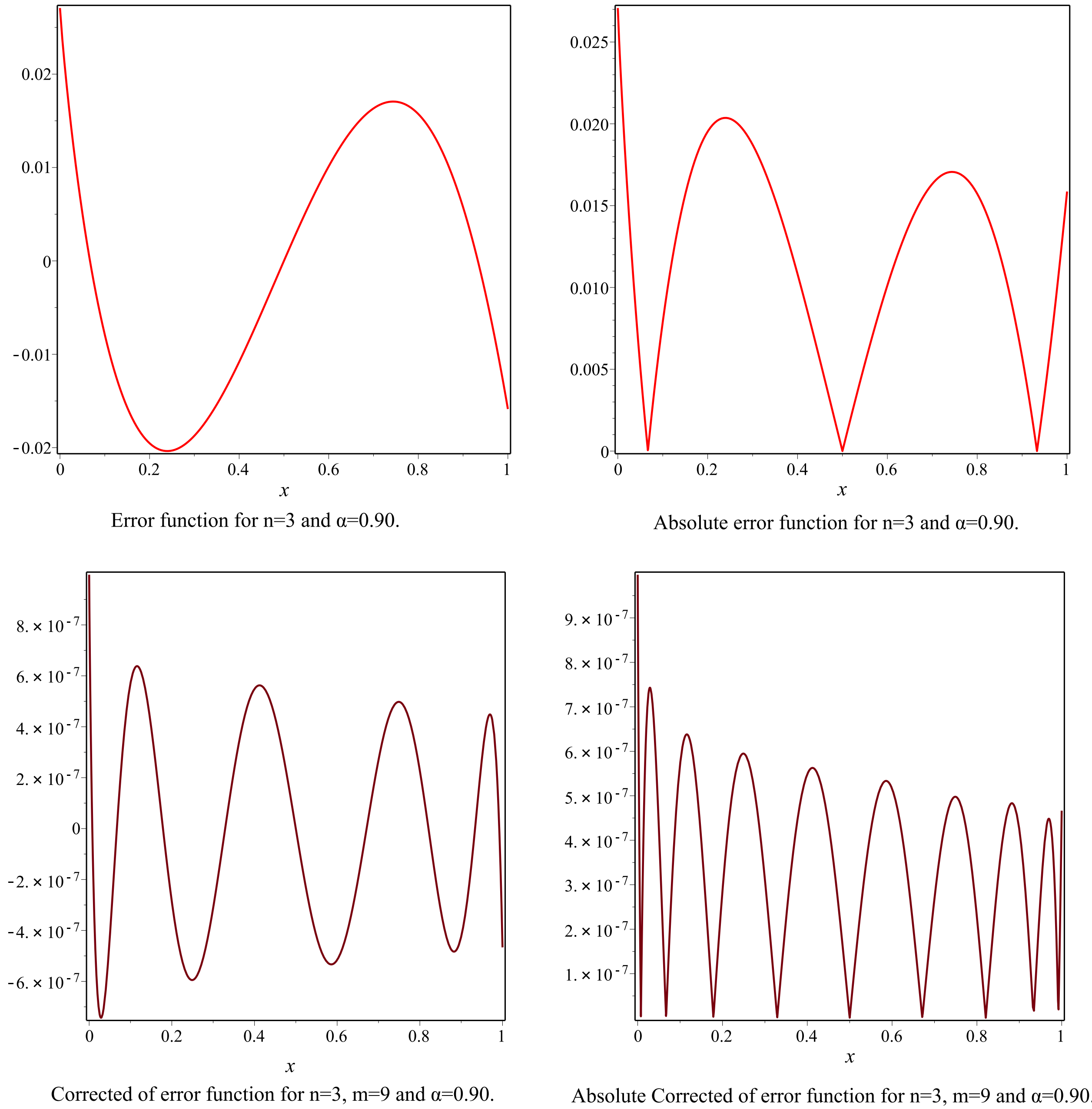


Figure 9 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 2, for the case $n = 3, m = 9$ and $\alpha = 0.90$.

where the vector $G^T = [g_0(x), g_1(x), \dots, g_n(x)]^T$, represents the non-homogenous term. By substituting the Eqs. (22)–(24), in Eq. (1) we obtain

$$C^{\mathrm{T}} D^{\alpha} \phi^*(x) = G^{\mathrm{T}} \phi^*(x) + q(x) C^{\mathrm{T}} \phi^*(x) + z(x) \left(C^{\mathrm{T}} \phi^*(x)\right)^r, \qquad (25)$$

it can be written as

$$C^{\mathrm{T}} D^{\alpha} \phi^*(x) - q(x) C^{\mathrm{T}} \phi^*(x) - z(x) \left(C^{\mathrm{T}} \phi^*(x)\right)^r - G^{\mathrm{T}} \phi^*(x) = 0, \qquad (26)$$

The interpolation error may be reduced by using the roots of Chebyshev polynomials

$$x_i = \frac{1}{2} + \frac{1}{2\cos\left((2i+1)\frac{\pi}{2n}\right)}, \quad i = 0, 1, \dots, n-1, \qquad (27)$$

By substituting these roots in Eq. (26), we obtain $n - 1$ of equations where the unknowns are c_i and each equation equals zero. Nonlinear equations can be solved by using Newton's iteration method. Consequently $y(x)$ which is given in Eq. (10) can be calculated, and this method (collocation method) is used to avoid the difficulty of integration.

4. Error analysis and estimation of the absolute error

In this section,we present the error analysis of the method used. Residual correction procedure which may estimate the absolute error will be assigned for the problem.

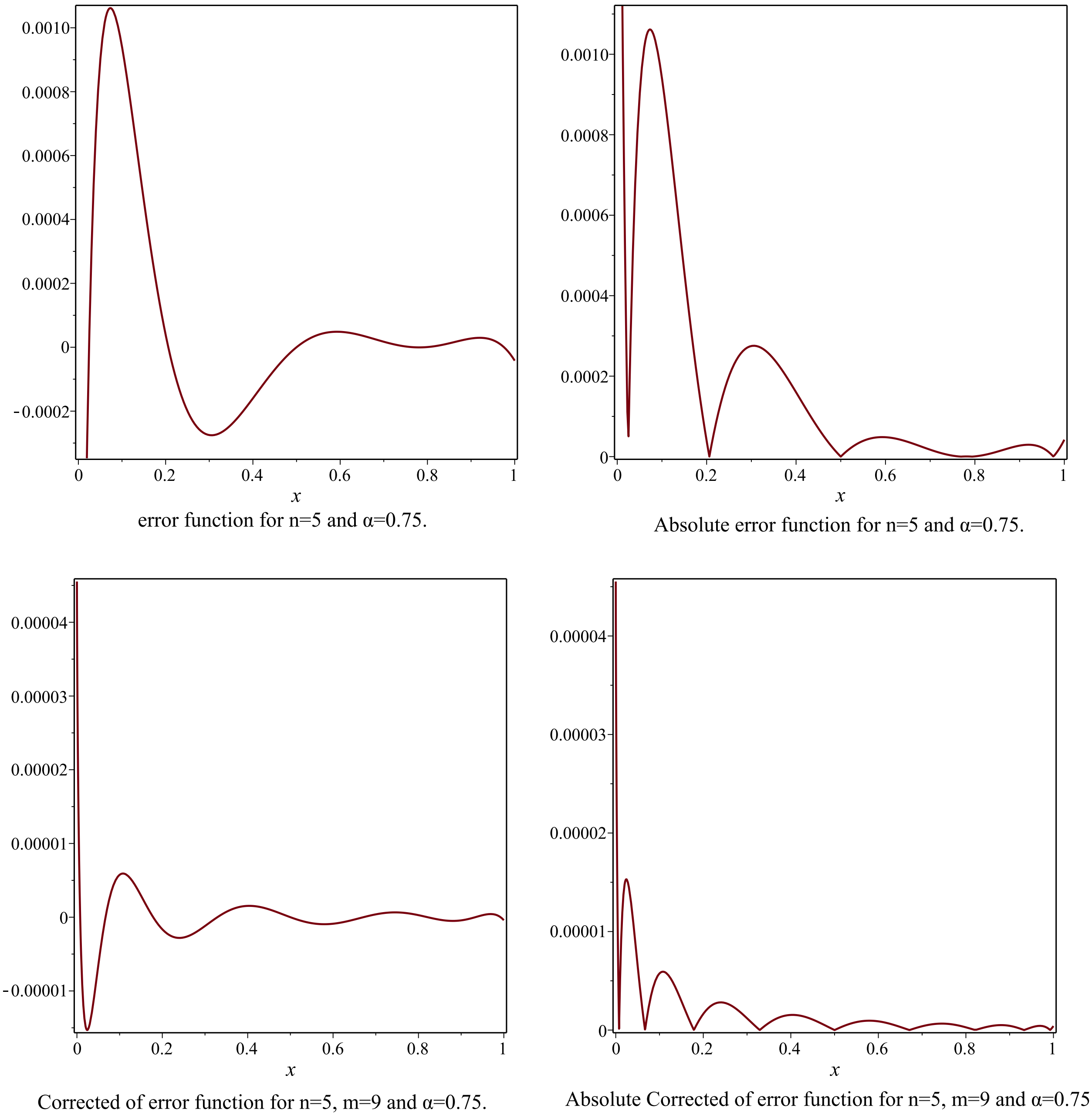


Figure 10 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 2, for the case $n = 5, m = 9$ and $\alpha = 0.75$.

Let $y_{n,\alpha}(x)$ and $y(x)$ be the approximate solution and the exact solution of (1), respectively. The following procedure, residual correction can be assigned to the estimation of the absolute error (Celik, 2006).

First, adding and subtracting the term

$$R := D^{\alpha}y_{n,\alpha}(x) + q(x)y_{n,\alpha}(x) + z(x)(y_{n,\alpha}(x))^{r} + g(x), \qquad (28)$$

to (1) yield the following differential equation

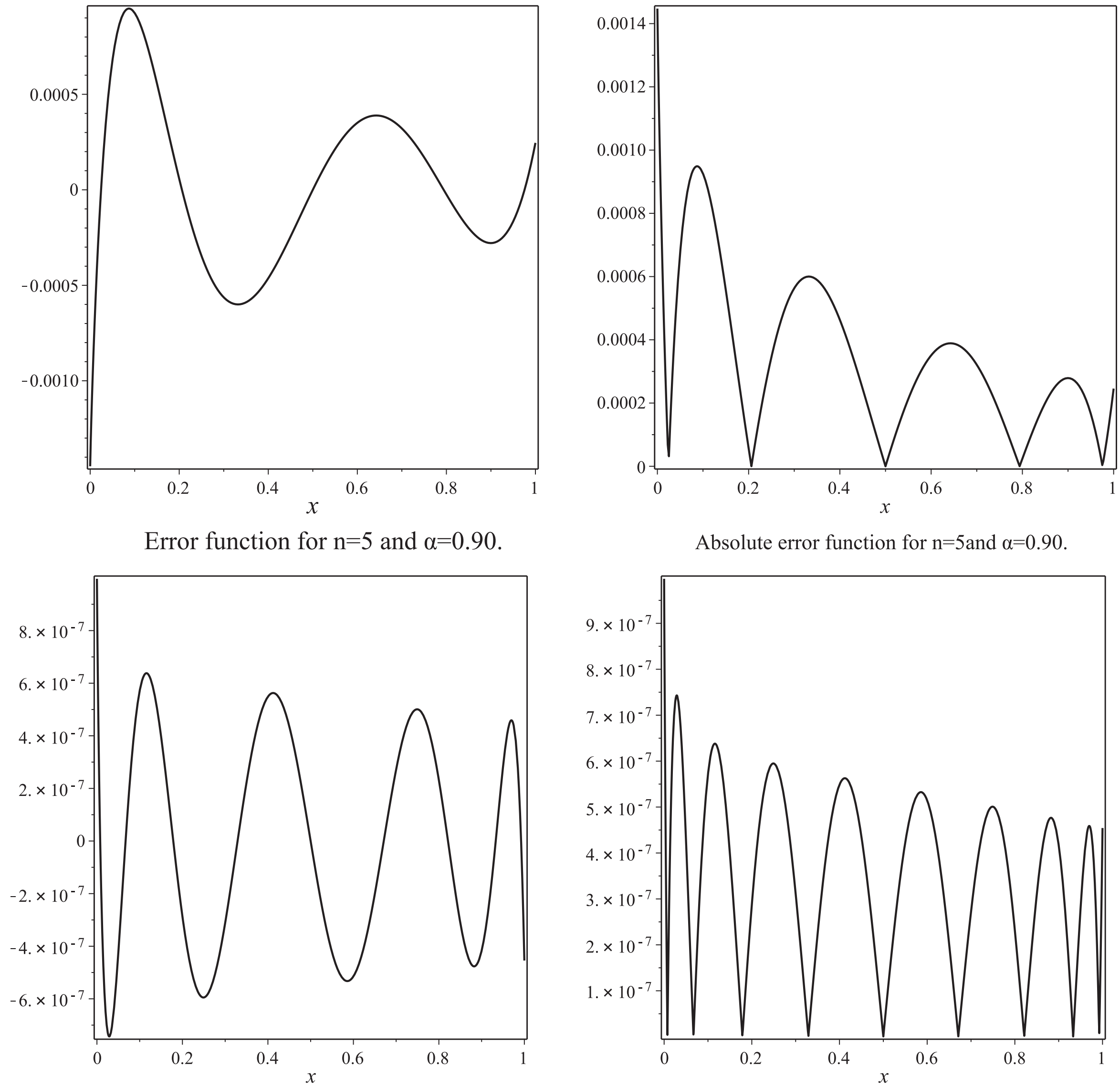
$$D^{\alpha}e_{n,\alpha}(x) + q(x)e_{n,\alpha}(x) + z(x)(e_{n,\alpha}(x))^{r} = g(x) - R, \qquad (29)$$

with the conditions

$$e(0) = 0, \quad e(1) = 0, \quad \text{or} \quad e'(0) = 0, \quad e(0) = 0. \qquad (30)$$

where $e_n(x) = y(x) - y_{n,\alpha}(x)$. For a given value m let $e^*_m(x)$ be the approximate solution of (29), where m is a polynomials degree of (29) and $m \geqslant n$.

Corollary 1. *Let $y_{n,\alpha}(x)$ be the approximate solution of (1) and $e^*_m(x)$ is the approximate solution of (29). Then $y_{n,\alpha}(x) + e^*_m(x)$*



Error function for n=5 and α=0.90.

Absolute error function for n=5and α=0.90.

Corrected of error function for n=5, m=9 and α=0.90.

Absolute Corrected of error function for n=5, m=9 and α=0.90.

Figure 11 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 2, for the case $n = 5, m = 9$ and $\alpha = 0.90$.

is also an approximate solution of (1) *and its error functions is* $e_{n,\alpha}(x) - e^*_m(x)$.

We term the approximate solution $y_{n,\alpha}(x) + e^*_m(x)$ *as the corrected approximate solution. Note that if* $\|e_n(x) - e^*_m(x)\| < \epsilon$, *then the absolute error can be estimated by* $e^*_m(x)$. *Moreover, if* $\|e_{n,\alpha}(x) - e^*_m(x)\| < \|y(x) - y_{n,\alpha}(x)\|$, *then* $y_{n,\alpha}(x) + e^*_m(x)$ *is a more accurate solution than* $y_{n,\alpha}(x)$ *in any given norm.*

5. Illustrative examples

To illustrate the effectiveness of the presented method, the following examples of linear and non-linear fractional differential equations (FDEs) are provided.

Example 1. Consider the fractional differential equation which has been considered by Saadatmandi and Dehghan (2010).

$$D^\alpha y(x) + y(x) = 0, \quad y(0) = 1, \quad 0 < \alpha < 1. \tag{31}$$

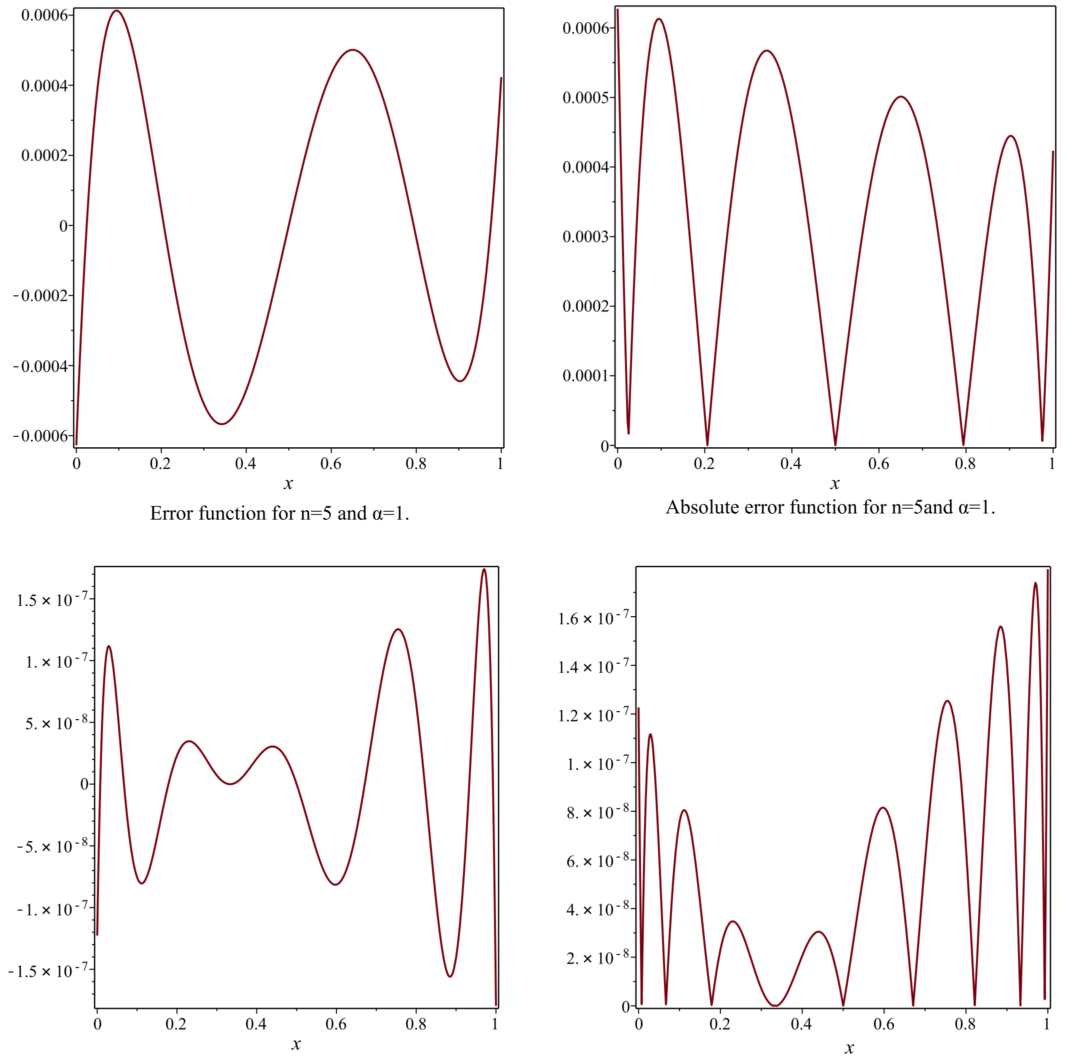
the exact solution of this problem is

$$y(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)}.$$

In this problem $q(x) = -1, z(x) = 0, g(x) = 0$ and $r = 1$, we approximate the solution as

$$y_{n,\alpha}(x) = \sum_{k=0}^{n} c_k B^\alpha_{n,k}(x - c) = C^T \phi^*(x). \tag{32}$$

By applying the method which is developed in Section 3 with $n = 3$ and $n = 5$ for $\alpha = 0.75, \alpha = 0.9$ and $\alpha = 1$, a good result



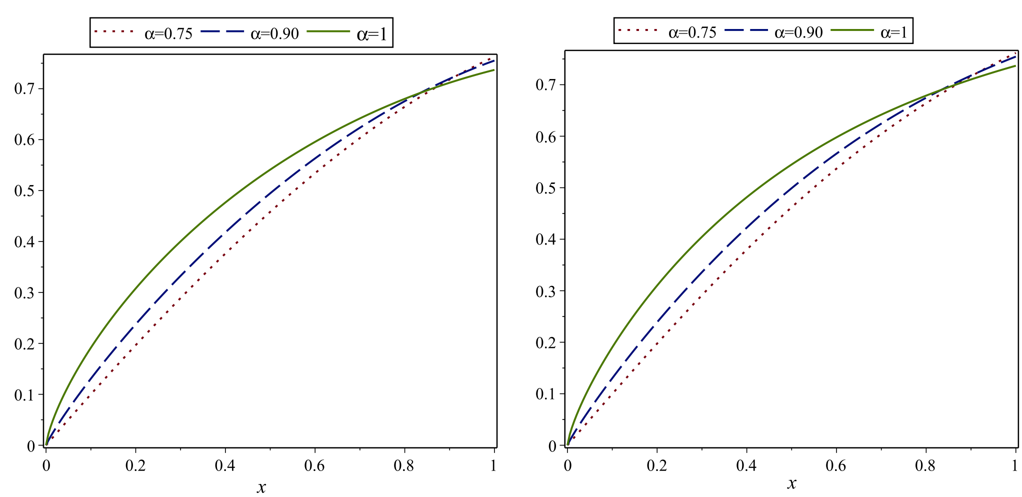
Error function for n=5 and α=1.

Absolute error function for n=5and α=1.

Corrected of error function for n=5, m=9 and α=1.

Absolute Corrected of error function for n=5, m=9 and α=1.

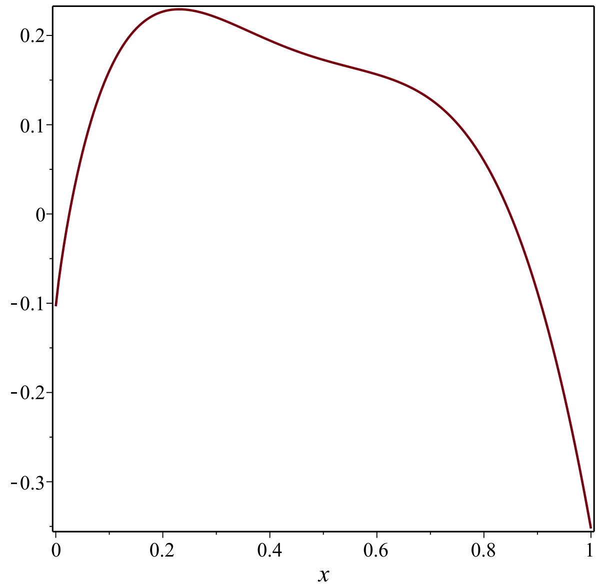
Figure 12 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 2, for the case $n = 5, m = 9$ and $\alpha = 1$.



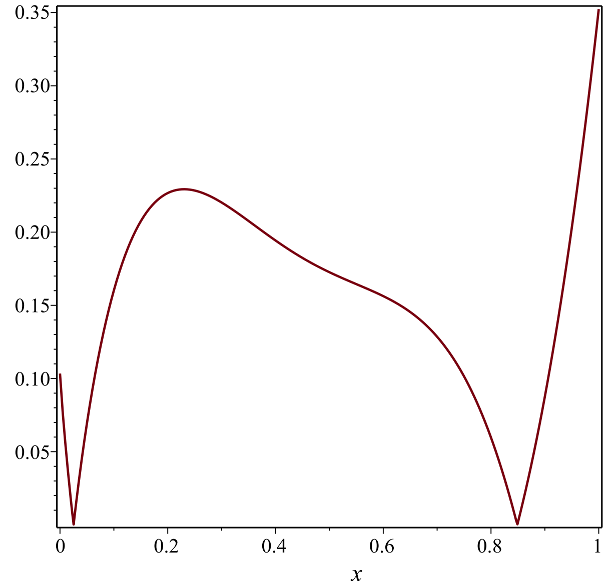
Compare between α=0.75,α=0.90 and α=1 to Example 2, for the case n=3

Compare between α=0.75,α=0.90 and α=1 to Example 2, for the case n=5

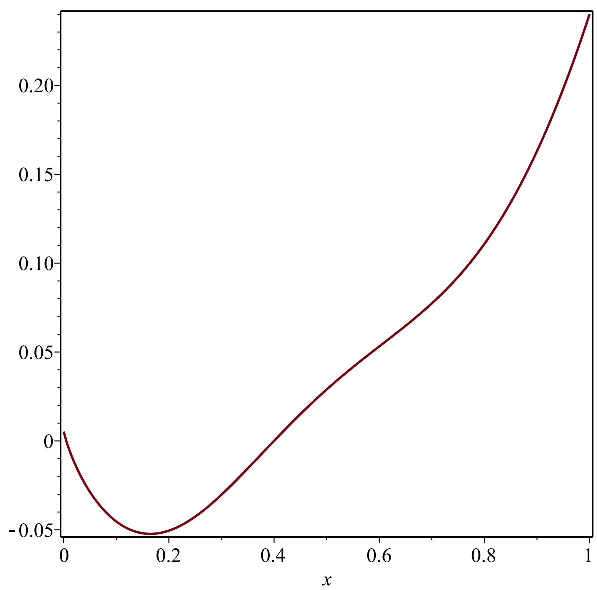
Figure 13 Compare between $\alpha = 0.75, \alpha = 0.90$ and $\alpha = 1$ to Examples 2, for the case $n = 3$, and $n = 5$.



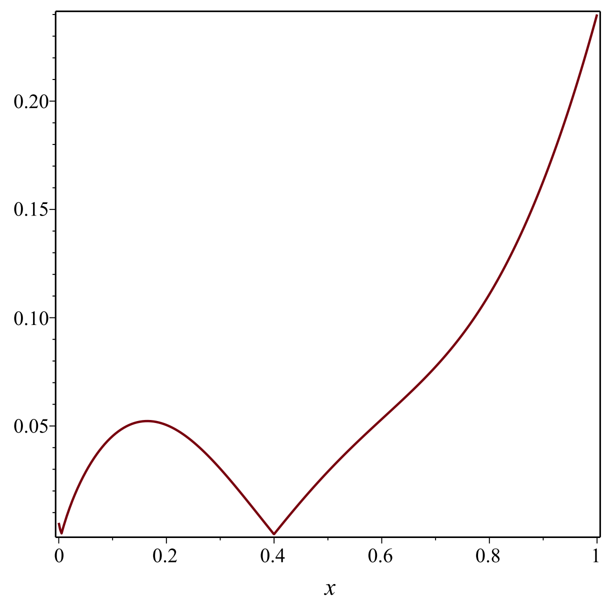
Error function for n=3 and α=0.90.



Absolute error function for n=3 and α=0.90.



Corrected of error function for n=3, m=9 and α=0.90.



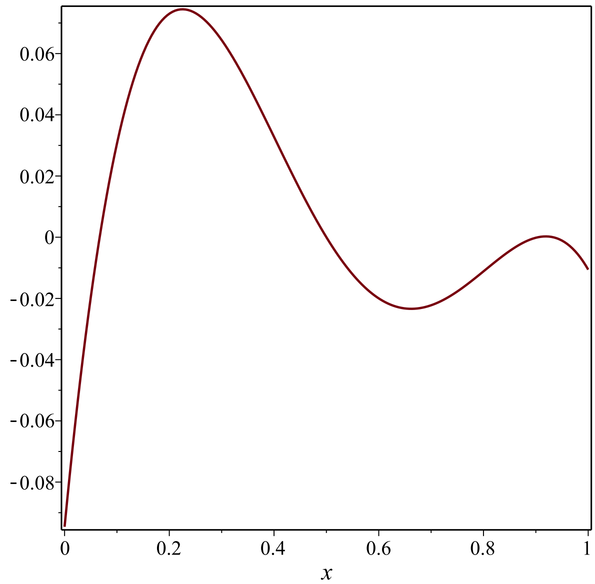
Absolute Corrected of error function for n=3, m=9 and α=0.90.

Figure 14 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 3, for the case $n = 3, m = 9$ and $\alpha = 0.90$.

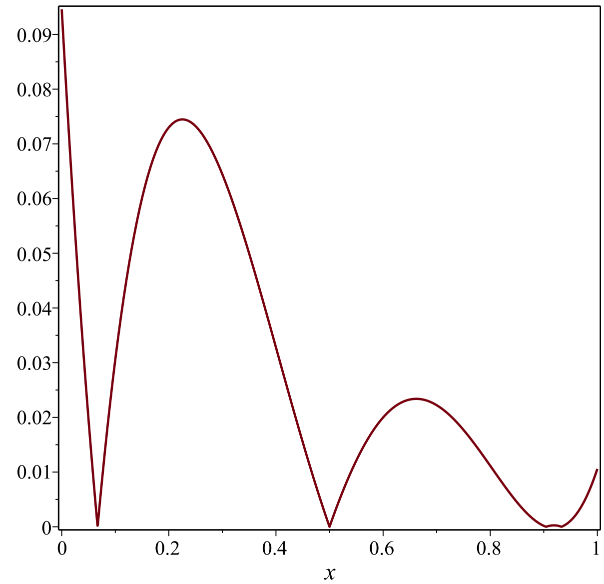
with a tiny error is obtained. We use the method in Section 4 to estimate the error and correct our approximate solution. The absolute errors and their estimations obtained from residual correction procedure for $m = 9$ are plotted in Figs. 1–6. In addition, the corrected approximate solutions $y_{n,\alpha}(x) + e_m^*(x)$ are represented in the same figures. It is noticed from the figures that residual correction procedure works well and the corrected approximate solutions are better than the approximate solutions. Note that when $\alpha = 1$, the exact solution is given as $y(x) = exp(-x)$. Comparisons between $\alpha = 0.75, \alpha = 0.90$ and $\alpha = 1$ to Examples 1, for the case $n = 3$, and $n = 5$ are plotted in Fig. 7.

Example 2. Consider the nonlinear fractional differential equation which has been considered by Yüzbası (2013).

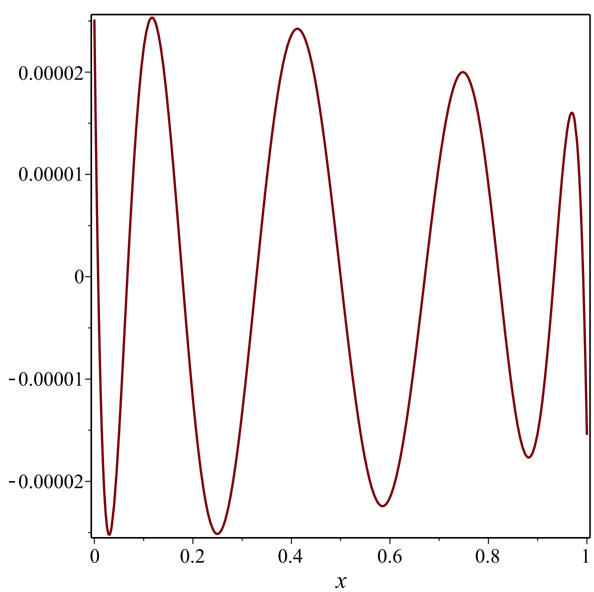
$$\frac{d^\alpha y(x)}{dx^\alpha} + y^2(x) = 1, \quad y(0) = 0, \quad 0 < \alpha < 1. \tag{33}$$



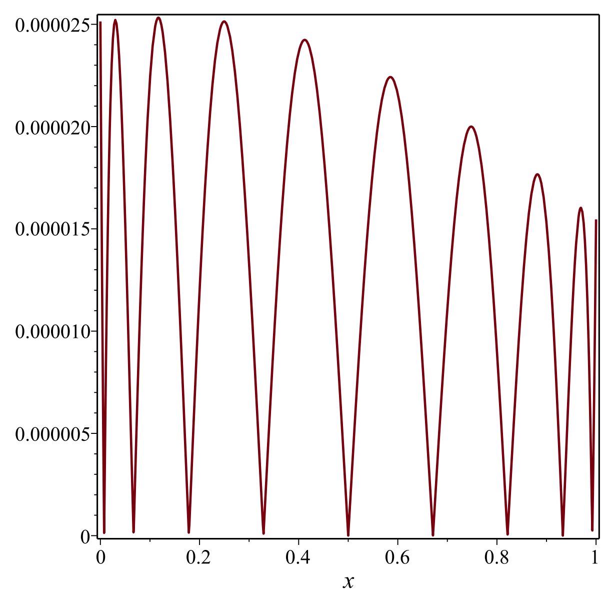
Error function for n=3 and α=1.



Absolute error function for n=3 and α=1.



Corrected of error function for n=3, m=9 and α=1.



Absolute Corrected of error function for n=3, m=9 and α=1.

Figure 15 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 3, for the case $n = 3, m = 9$ and $\alpha = 1$.

In this problem $q(x) = 0, z(x) = -1, g(x) = 1$ and $r = 2$, by applying the method developed in Section 3 with $n = 3$ and $n = 5$ for $\alpha = 0.75, \alpha = 0.9$ and $\alpha = 1$, good result is obtained with a slight error. The method is used in Section 4 to estimate the error and correct our approximate solution. In this example there is no exact solution, so we find error function and the corrected error function by residual correction procedure for $m = 9$, and all the results are plotted in Figs. 8–12. Comparisons between $\alpha = 0.75, \alpha = 0.90$ and $\alpha = 1$ to Examples 1, for the case $n = 3$, and $n = 5$ are plotted in Fig. 13.

Example 3. Consider the nonlinear fractional differential equation which has been considered by Yüzbası (2013).

$$\frac{d^{\alpha}y(x)}{dx^{\alpha}} = 2y(x) - y^2(x) + 1, \quad y(0) = 0, \quad 0 < \alpha < 1. \tag{34}$$

The exact solution for the problem for $\alpha = 1$ is given by

$$y(x) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{35}$$

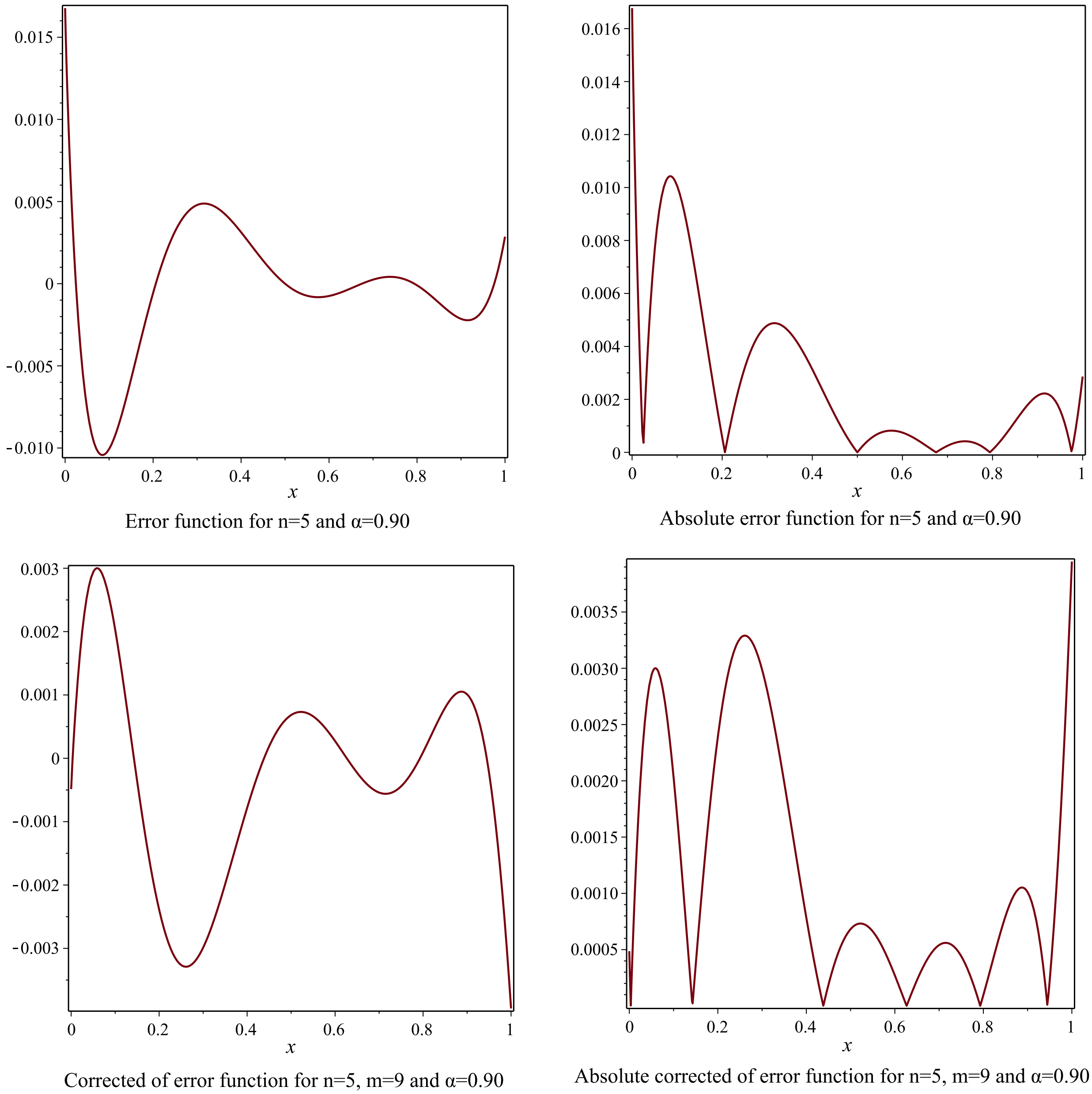


Figure 16 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 3, for the case $n = 5, m = 9$ and $\alpha = 0.90$.

In this problem $q(x) = 2, z(x) = 1, g(x) = 1$ and $r = 2$, we approximate the solution as

$$y_{n,\alpha}(x) = \sum_{k=0}^{n} c_k B^{\alpha}_{n,k}(x - c) = C^T \phi^*(x). \tag{36}$$

We present the solution with $n = 3$ and $n = 5$ for $\alpha = 0.9$ and $\alpha = 1$. We use the method in Section 4 to estimate the error and correct our approximate solution. In this example there is no exact solution, so we find the error function and the corrected error function by residual correction procedure for $m = 9$. All the results are plotted in Figs. 14–17. Comparisons between $\alpha = 0.75, \alpha = 0.90$ and $\alpha = 1$ to Examples 1, for the case $n = 3$, and $n = 5$ are plotted in Fig. 18.

6. Conclusions

In this paper, the Bernstein operational matrix of derivative was applied to solve linear and non-linear fractional differential equations. Different from other numerical techniques, the few D^{α} matrices are obtained. These matrices are used to approximate the numerical solution of fractional differential equations. It can be clearly noticed that the proposed method performs well even on a few number of terms of the Bernstein polynomials. The method is presented with some error analysis, and residual correction procedure is extended for this problem. The subjects of our future works can be exemplified by applying the presented technique for solving system of frac-

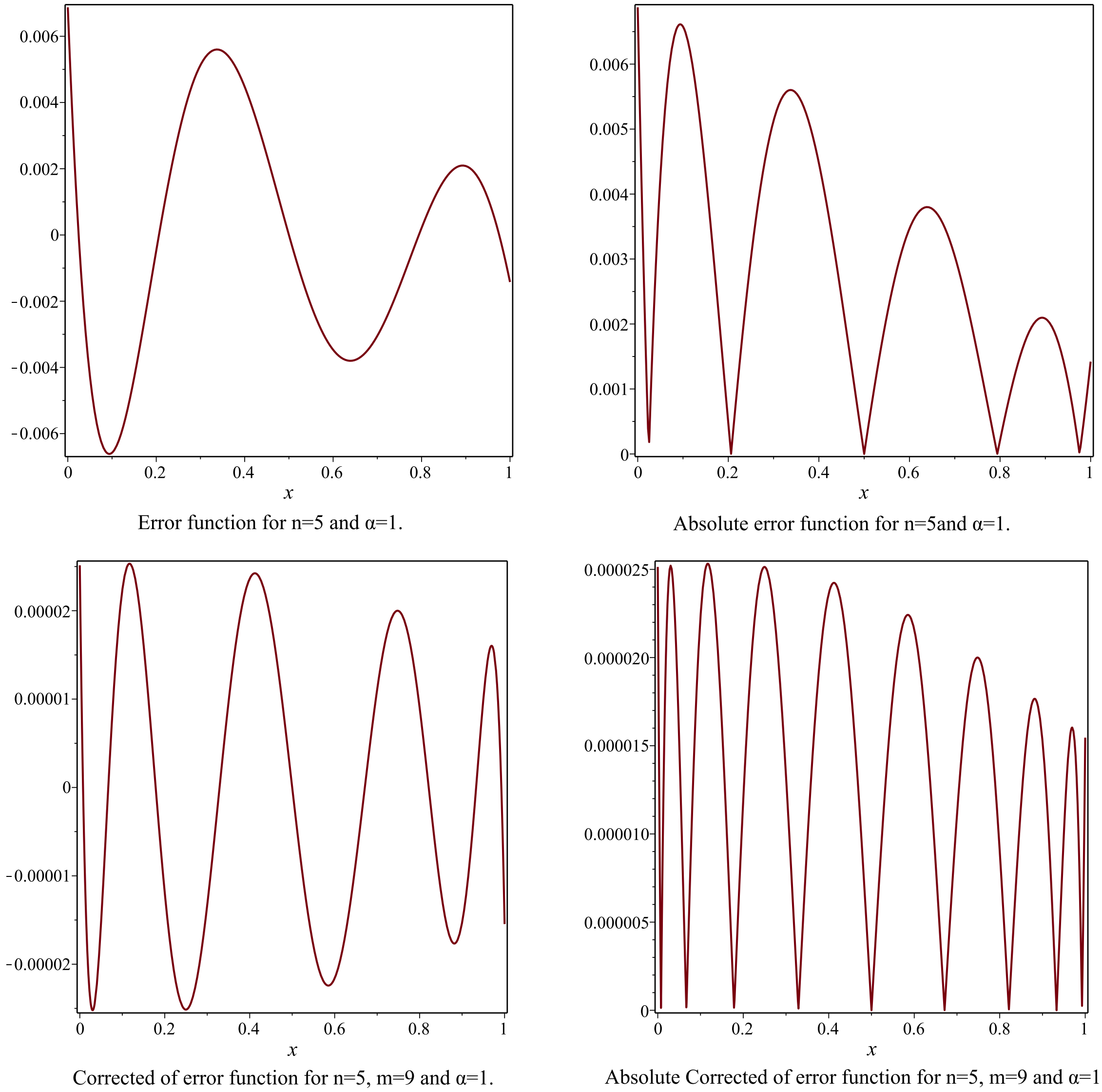


Figure 17 The error function, the absolute of error function, the corrected error function and the absolute corrected error function to Example 3, for the case $n = 5, m = 9$ and $\alpha = 1$.

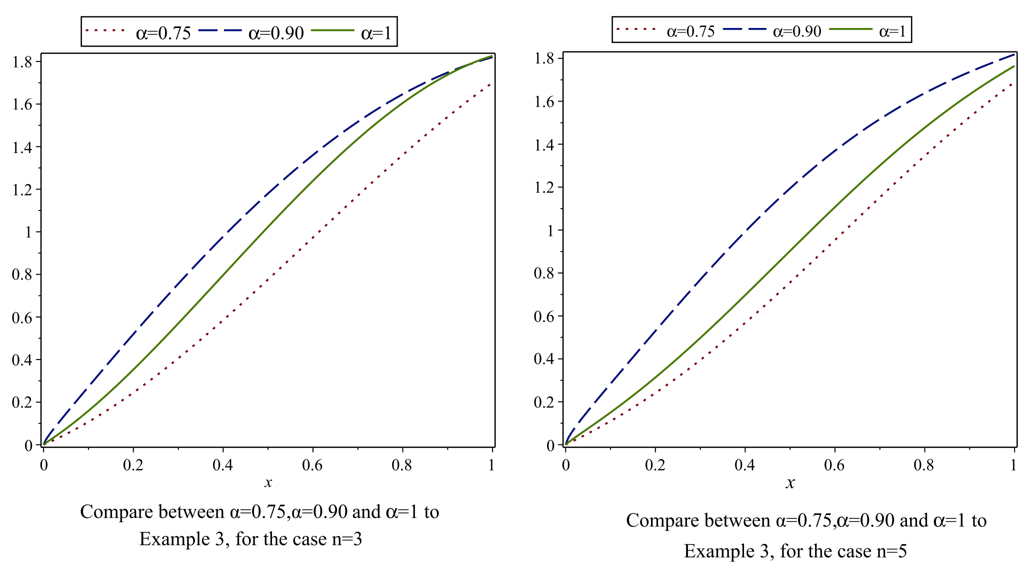


Figure 18 Compare between $\alpha = 0.75, \alpha = 0.90$ and $\alpha = 1$ to Examples 3, for the case $n = 3$, and $n = 5$.

tional differential equations and fractional partial differential equations.

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